Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A Theorem in Determinants.

By W. H. Metzler, Ph. D.

The theorem in question is an extension of one given by Binet in Journ. de l'Ec. Polyt. IX, cah. 16, pp. 280–302.*

If we have the four sets of n quantities

$$\begin{matrix} a_{11} & a_{12} & a_{13} \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots\ldots a_{3n} \\ a_{41} & a_{42} & a_{43} \ldots\ldots a_{4n}, \end{matrix}$$

and $\sum A_{\substack{1234\\1234}}$† denotes the sum of the n_4 determinants of the fourth order obtainable from them, $\sum A_{\substack{123\\123}}$ denotes the like sum obtainable from the first three sets, etc. Then Binet proved that

$$\sum a_{11} \sum A_{\substack{23\\23}} + \sum a_{21} \sum A_{\substack{13\\23}} + \sum a_{31} \sum A_{\substack{12\\23}} = \sum A_{\substack{123\\123}}$$

and

$$\sum a_{11} \sum A_{\substack{234\\234}} + \sum a_{21} \sum A_{\substack{134\\234}} + \sum a_{31} \sum A_{\substack{124\\234}} + \sum a_{41} \sum A_{\substack{123\\234}} = 0 .$$

These may be extended to the case where instead of four we have m sets of n quantities, and instead of taking the sets one and $m - 1$ at a time, we take them l and $m - l$ at a time.

The proof of the general case will be better understood if we first consider a special case.

---

* Vide Muir's "The Theory of Determinants in the Historical Order of its Development," Part I, pp. 86-91.

† It is to be observed that the notation used throughout this paper is inclusive, i. e. $A_{\substack{rst\ldots\\uvw\ldots}}$ denotes the minor formed by the constituents in the intersection of the $r, s, t \ldots\ldots$ rows with the $u, v, w \ldots\ldots$ columns.

If

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{vmatrix}$$

then

$$\begin{aligned}
&\Big(A_{\substack{12\\12}} + A_{\substack{12\\13}} + \dots + A_{\substack{12\\56}}\Big)\Big(A_{\substack{34\\12}} + A_{\substack{34\\13}} + \dots + A_{\substack{34\\56}}\Big)\\
-&\Big(A_{\substack{13\\12}} + A_{\substack{13\\13}} + \dots + A_{\substack{13\\56}}\Big)\Big(A_{\substack{24\\12}} + A_{\substack{24\\13}} + \dots + A_{\substack{24\\56}}\Big)\\
+&\Big(A_{\substack{14\\12}} + A_{\substack{14\\13}} + \dots + A_{\substack{14\\56}}\Big)\Big(A_{\substack{23\\12}} + A_{\substack{23\\13}} + \dots + A_{\substack{23\\56}}\Big)\\
+&\Big(A_{\substack{23\\12}} + A_{\substack{23\\13}} + \dots + A_{\substack{23\\56}}\Big)\Big(A_{\substack{14\\12}} + A_{\substack{14\\13}} + \dots + A_{\substack{14\\56}}\Big)\\
-&\Big(A_{\substack{24\\12}} + A_{\substack{24\\13}} + \dots + A_{\substack{24\\56}}\Big)\Big(A_{\substack{13\\12}} + A_{\substack{13\\13}} + \dots + A_{\substack{13\\56}}\Big)\\
+&\Big(A_{\substack{34\\12}} + A_{\substack{34\\13}} + \dots + A_{\substack{34\\56}}\Big)\Big(A_{\substack{12\\12}} + A_{\substack{12\\13}} + \dots + A_{\substack{12\\56}}\Big)\\
=&\, 2\Big(A_{\substack{1234\\1234}} + A_{\substack{1234\\1235}} + A_{\substack{1234\\1236}} + A_{\substack{1234\\1245}} + A_{\substack{1234\\1246}}\\
+&\quad A_{\substack{1234\\1256}} + A_{\substack{1234\\1345}} + A_{\substack{1234\\1346}} + A_{\substack{1234\\1356}} + A_{\substack{1234\\1456}}\\
+&\quad A_{\substack{1234\\2345}} + A_{\substack{1234\\2346}} + A_{\substack{1234\\2356}} + A_{\substack{1234\\2456}} + A_{\substack{1234\\3456}}\Big).
\end{aligned}$$

For if we expand the left-hand member, the sum of corresponding terms in the products vanishes or not according as the factors of the terms have or have not a column in common, i. e. according as there is or is not a number common to the lower lines of the suffixes of both factors.

Thus

$$A_{\substack{12\\12}}A_{\substack{34\\12}} - A_{\substack{13\\12}}A_{\substack{24.\\12}} + A_{\substack{14\\12}}A_{\substack{23\\12}} + A_{\substack{23\\12}}A_{\substack{14\\12}} - A_{\substack{24\\12}}A_{\substack{13\\12}} + A_{\substack{34\\12}}A_{\substack{12\\12}} = 0,$$

$$A_{\substack{12\\12}}A_{\substack{34\\13}} - A_{\substack{13\\12}}A_{\substack{24\\13}} + A_{\substack{14\\12}}A_{\substack{23\\13}} + A_{\substack{23\\12}}A_{\substack{14\\13}} - A_{\substack{24\\12}}A_{\substack{13\\13}} + A_{\substack{34\\12}}A_{\substack{12\\13}} = 0,$$

etc.

$$A_{\substack{12\\12}}A_{\substack{34\\34}} - A_{\substack{13\\12}}A_{\substack{24\\34}} + A_{\substack{14\\12}}A_{\substack{23\\34}} + A_{\substack{23\\12}}A_{\substack{14\\34}} - A_{\substack{24\\12}}A_{\substack{13\\34}} + A_{\substack{34\\12}}A_{\substack{12\\34}} = A_{\substack{1234\\1234}},$$

$$A_{\substack{12\\13}}A_{\substack{34\\24}} - A_{\substack{13\\13}}A_{\substack{24\\24}} + A_{\substack{14\\13}}A_{\substack{23\\24}} + A_{\substack{23\\13}}A_{\substack{14\\24}} - A_{\substack{24\\13}}A_{\substack{13\\24}} + A_{\substack{34\\13}}A_{\substack{12\\24}} = -A_{\substack{1234\\1234}},$$

etc.

It is easily seen that $A_{\substack{1234\\1234}}$ as well as each of the other terms on the right will be obtained in $\frac{4\,.\,3}{1\,.\,2} = 6$ different ways, four of which will give a positive and two a negative sign, since in the set of combinations 1234, 1324, 1423, 2314, 2413, 3412 there are four having an even number of inversions and two having an odd number of inversions. Hence the truth of the equation.

In general if

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \dots\dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots\dots a_{2n} \\ \dots & \dots & \dots\dots\dots\dots \\ a_{m1} & a_{m2} & a_{m3} \dots\dots a_{mn} \\ \dots & \dots & \dots\dots\dots\dots \\ a_{n1} & a_{n2} & a_{n3} \dots\dots a_{nn} \end{vmatrix},$$

then

$$(-1)^{\nu} \sum_{\delta=1}^{\delta=n_l} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\delta}{l})}} \cdot \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{1}{l}) \\ (n\,|\,\underset{\delta}{m-l})}}$$

$$+ (-1)^{\nu} \sum_{\delta=1}^{\delta=n_l} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\delta}{l})}} \cdot \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{2}{l}) \\ (n\,|\,\underset{\delta}{m-l})}},$$

$$+ \text{ etc.}$$

$$= \phi(m,\, l)\,.\sum_{\beta=1}^{\beta=n_m} A^{*}_{\substack{(n\,|\,m) \\ (n\,|\,\underset{\beta}{\overset{\alpha}{m}})}},$$

or

$$\sum_{\gamma=1}^{\gamma=m_l} (-1)^{\nu} \left[\sum_{\delta=1}^{\delta=n} A_{\substack{(n\,|\,\underset{\alpha}{m}\,|\,\underset{\gamma}{l}) \\ (n\,|\,\underset{\delta}{l})}} \sum_{\delta=1}^{\delta=n_{m-l}} A_{\substack{(n\,|\,\underset{\alpha}{\overline{m}}\,|\,\underset{\gamma}{l}) \\ (n\,|\,\underset{\delta}{m-l})}} \right]$$

$$= \phi(m,\, l) \sum_{\beta=1}^{\beta=n_m} A_{\substack{(n\,|\,m) \\ (n\,|\,\underset{\beta}{\overset{\alpha}{m}})}},$$

where ν denotes the number of inversions in $(n|\underset{\alpha}{m}|\underset{\gamma}{l})(n|\underset{\alpha}{\overline{m}}|\underset{\gamma}{l})$,* and $\phi(m,\, l)$ denotes the excess of the number of combinations in the set

$$(n|\underset{\beta}{m}|\underset{1}{l})(n|\underset{\beta}{\overline{m}}|\underset{1}{l}),\ (n|\underset{\beta}{m}|\underset{2}{l})(n|\underset{\beta}{\overline{m}}|\underset{2}{l}),\ \dots\dots\ (n|\underset{\beta}{m}|\underset{\delta}{l})(n|\underset{\beta}{\overline{m}}|\underset{\delta}{l}),\quad \delta = n_l$$

which have an even number of inversions over those which have an odd number.*

---

* For an explanation of this notation vide this Journal, vol. XX, No. 3.

For if we expand the left-hand member as before, it is easily seen that the sum of corresponding terms of the products will vanish or not according as $(n \mid \underset{\delta}{l})$ and $(n \mid \underset{\delta}{m - l})$ have or have not any numbers in common. Any term $A_{\substack{(n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\beta}{m})}}$ which does not vanish will be obtained in m_l ways from those particular sums of corresponding terms of products which are given by

$$\sum_{\gamma=1}^{\gamma=m_l}(-1)^{\gamma}\left[A_{\substack{(n \mid \underset{\alpha}{m} \mid \underset{\gamma}{l}) \\ (n \mid \underset{\beta}{m} \mid \underset{\delta}{l})}} \cdot A_{\substack{(n \mid \underset{\alpha}{m} \mid \underset{\gamma}{l}) \\ (n \mid \underset{\beta}{m} \mid \underset{\delta}{l})}}\right], \quad (\delta = 1,\ 2,\ \ldots,\ m_l).$$

The coefficient of $A_{\substack{(n \mid \underset{\alpha}{m}) \\ (n \mid \underset{\beta}{m})}}$ is therefore $\phi\,(m,\ l)$.*

The theorem is thus established.

SYRACUSE UNIVERSITY, *February* 10, 1898.

---

* Vide author's paper read before the Am. Math. Soc., Feb. 26, 1898, and to appear in a later number of this Journal.